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CHARACTERIZATION OF JOINT ERGODICITY FOR NON-COMMUTING TRANSFORMATIONS

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ABSTRACT

The study of jointly ergodic transformations, begun in [2] and [1], is continued. The main result is that, if $T_1, T_2, \ldots, T_s$ are arbitrary measure preserving transformations of a probability space $(X, \mathcal{B}, \mu)$, then

$$\frac{1}{N}\sum_{n=0}^{N-1} T_1^n f_1 \cdot T_2^n f_2 \cdot \cdots \cdot T_s^n f_s \xrightarrow[N\to\infty]{L^2} \int_X f_1 d\mu \cdot \int_X f_2 d\mu \cdot \cdots \cdot \int_X f_s d\mu,$$

$$f_1, f_2, \ldots, f_s \in L^\infty(X, \mathcal{B}, \mu),$$

if and only if the following conditions are satisfied:

(1) $T_1 \times T_2 \times \cdots \times T_s$ is ergodic.

(2) $(1/N)\sum_{n=0}^{N-1} \int_X T_1^n f_1 \cdot T_2^n f_2 \cdot \cdots \cdot T_s^n f_s d\mu \to_{N\to\infty} \int_X f_1 d\mu \cdot \cdots \cdot \int_X f_s d\mu$, $f_1, f_2, \ldots, f_s \in L^\infty(X, \mathcal{B}, \mu)$.

1. Introduction

Let $(X, \mathcal{B}, \mu)$ be a probability space and $T_1, T_2, \ldots, T_s$ measure preserving transformations thereof. $T_1, T_2, \ldots, T_s$ are called *jointly ergodic* (resp. *w-jointly ergodic*) if

$$(1.1) \qquad \frac{1}{N}\sum_{n=0}^{N-1} T_1^n f_1 \cdot T_2^n f_2 \cdot \cdots \cdot T_s^n f_s \xrightarrow[N\to\infty]{} \int_X f_1 d\mu \cdot \int_X f_2 d\mu \cdot \cdots \cdot \int_X f_s d\mu$$

in L^2-norm (resp. weakly in L^2) for every $f_1, f_2, \ldots, f_s \in L^\infty(X, \mathcal{B}, \mu)$. Similarly, $T_1, T_2, \ldots, T_s$ are *uniformly jointly ergodic* or *uniformly w-jointly ergodic* if (1.1) is replaced by

$$\frac{1}{N-M}\sum_{n=M}^{N-1} T_1^n f_1 \cdot T_2^n f_2 \cdot \cdots \cdot T_s^n f_s \xrightarrow[N-M\to\infty]{} \int_X f_1 d\mu \cdot \int_X f_2 d\mu \cdot \cdots \cdot \int_X f_s d\mu$$

(see [1, Def. 2.1, Th. 2.1]).

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It was proved in [2, Th. 3.1] that for invertible commuting transformations all the above notions coincide and, moreover, are equivalent to the transformations $T_1 \times T_2 \times \cdots \times T_s$ and $T_i T_j^{-1}$, $1 \leqq i < j \leqq s$, being ergodic. A version of this result in case $T_1, T_2, \ldots, T_s$ are not necessarily invertible is given in [1, Th. 2.3].

In this paper we shall characterize joint ergodicity in the general case, namely where the transformations are not necessarily commuting. It turns out (see Theorem 2.1 *infra*) that joint ergodicity and w-joint ergodicity are still equivalent, and can be characterized by a condition generalizing the one given for commuting transformations. An analogue holds for uniform joint ergodicity. These results are the contents of Section 2.

Section 3 deals with the case of endomorphisms of compact abelian groups. The former results yield a simple characterization of joint ergodicity, and we use this to provide an example of two automorphisms which are jointly ergodic but not uniformly jointly ergodic,

2. The Characterization Theorem

Our main result is

THEOREM 2.1. *Let $T_1, T_2, \ldots, T_s$ be arbitrary measure preserving transformations of a probability space $(X, \mathscr{B}, \mu)$. The following conditions are equivalent*:

$$(1) \qquad \frac{1}{N}\sum_{n=0}^{N-1} T_1^n f_1 \cdot T_2^n f_2 \cdot \cdots \cdot T_s^n f_s \underset{N\to\infty}{\longrightarrow} \int_X f_1 d\mu \cdot \int_X f_2 d\mu \cdot \cdots \cdot \int_X f_s d\mu$$

strongly in $L^2(X, \mathscr{B}, \mu)$ for every $f_1, f_2, \ldots, f_s \in L^\infty(X, \mathscr{B}, \mu)$.

$$(2) \qquad \frac{1}{N}\sum_{n=0}^{N-1} T_1^n f_1 \cdot T_2^n f_2 \cdot \cdots \cdot T_s^n f_s \underset{N\to\infty}{\longrightarrow} \int_X f_1 d\mu \cdot \int_X f_2 d\mu \cdot \cdots \cdot \int_X f_s d\mu$$

weakly in $L^2(X, \mathscr{B}, \mu)$ for every $f_1, f_2, \ldots, f_s \in L^\infty(X, \mathscr{B}, \mu)$.

(3) (a) *$T_1 \times T_2 \times \cdots \times T_s$ is ergodic.*

$$\text{(b)} \quad (1/N)\sum_{n=0}^{N-1} \int_X T_1^n f_1 \cdot T_2^n f_2 \cdot \cdots \cdot T_s^n f_s d\mu \underset{N\to\infty}{\longrightarrow} \int_X f_1 d\mu \cdot \int_X f_2 d\mu \cdot \cdots \cdot \int_X f_s d\mu$$

for every $f_1, f_2, \ldots, f_s \in L^\infty(X, \mathscr{B}, \mu)$.

PROOF. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3(b)) are trivial, and (2) $\Rightarrow$ (3(a)) is proved just as Theorem 2.2 in [2].

(3) $\Rightarrow$ (1): Given $f_1, f_2, \ldots, f_s \in L^\infty(X, \mathscr{B}, \mu)$, we have to show that

$$\frac{1}{N}\sum_{n=0}^{N-1} T_1^n f_1 \cdot T_2^n f_2 \cdot \cdots \cdot T_s^n f_s \underset{N\to\infty}{\longrightarrow} \int_X f_1 d\mu \cdot \int_X f_2 d\mu \cdot \cdots \cdot \int_X f_s d\mu$$

strongly in $L^2(X,\mathscr{B},\mu)$. It may be assumed that each f_i is a real function bounded by 1 in absolute value, and that $\int_X f_1 d\mu = 0$. We shall use the following inequality, known as the van der Corput inequality (cf. [5, p. 25]): If $u_1, u_2, \ldots, u_N$ are complex numbers and $H \leqq N$ a positive integer, then

$$H^2\left|\sum_{n=1}^{N} u_n\right|^2 \leqq H(N+H-1)\sum_{n=1}^{N}|u_n|^2 + 2(N+H-1)\sum_{h=1}^{N-1}(H-h)\cdot \mathrm{Re}\sum_{n=1}^{N-h} u_n\bar{u}_{n+h}$$

(Re z denoting the real part of a complex number z).

Substituting $u_n = f_1(T_1^{n-1}x)f_2(T_2^{n-1}x)\cdots f_s(T_s^{n-1}x)$, $1 \leqq n \leqq N$, for any $x \in X$, and integrating both sides with respect to μ, we easily obtain

$$\left\|\frac{1}{N}\sum_{n=0}^{N-1} T_1^n f_1 \cdot T_2^n f_2 \cdot \cdots \cdot T_s^n f_s\right\|_2^2 \tag{2.1}$$
$$\leqq \frac{N+H-1}{HN^2}\cdot N + 2\sum_{h=1}^{H-1}\frac{(N+H-1)(H-h)(N-h)}{H^2N^2} S_{N,h}$$

where

$$S_{N,h} = \frac{1}{N-h}\sum_{n=0}^{N-h-1}\int_X T_1^n f_1 \cdot T_2^n f_2 \cdot \cdots \cdot T_s^n f_s \cdot T_1^{n+h} f_1 \cdot T_2^{n+h} f_2 \cdot \cdots \cdot T_s^{n+h} f_s d\mu. \tag{2.2}$$

From the given condition we see that for any fixed h

$$S_{N,h} \xrightarrow[N\to\infty]{} \int_X f_1 \cdot T_1^h f_1 d\mu \cdot \int_X f_2 \cdot T_2^h f_2 d\mu \cdot \cdots \cdot \int_X f_s \cdot T_s^h f_s d\mu.$$

Denote $(X^s, \mathscr{B}^s, \mu^s, T) = (X,\mathscr{B},\mu,T_1)\times(X,\mathscr{B},\mu,T)_2\times\cdots\times(X,\mathscr{B},\mu,T_s)$. Let $F: X \to \mathbf{R}^s$ be defined by:

$$F(x_1, x_2, \ldots, x_s) = f_1(x_1)f_2(x_2)\cdots f_s(x_s), \qquad (x_1, x_2, \ldots, x_s) \in X^s.$$

Holding H fixed, it now follows from (2.1) and (2.2) that

$$\overline{\lim_{N\to\infty}}\left\|\frac{1}{N}\sum_{n=0}^{N-1} T_1^n f_1 \cdot T_2^n f_2 \cdot \cdots \cdot T_s^n f_s\right\|_2^2 \leqq \frac{1}{H} + 2\sum_{h=1}^{H-1}\frac{H-h}{H^2}\int_{X^s} F\cdot T^h F d\mu^s. \tag{2.3}$$

Since T is ergodic

$$\frac{1}{H} + 2\sum_{h=1}^{H-1}\frac{H-h}{H^2}\int_{X^s} F\cdot T^h F d\mu^s$$
$$= \frac{1}{H} + \frac{1}{H^2}\left(\sum_{i,j=0}^{H-1}\int_{X^s} T^iF\cdot T^jF d\mu^s - \sum_{i=0}^{H-1}\int_{X^s} T^iF\cdot T^iF d\mu^s\right) \tag{2.4}$$
$$\leqq \frac{1}{H} + \int_{X^s}\left(\frac{1}{H}\sum_{h=0}^{H-1} T^h F\right)^2 d\mu^s \xrightarrow[H\to\infty]{} 0.$$

From (2.3) and (2.4) we conclude that

$$\frac{1}{N}\sum_{n=0}^{N-1} T_1^n f_1 \cdot T_2^n f_2 \cdot \cdots \cdot T_s^n f_s \xrightarrow[N\to\infty]{} 0$$

strongly in $L^2(X, \mathscr{B}, \mu)$. This completes the proof.

REMARK 2.1. Using the terminology introduced in Section 1, we see that the equivalence of the first two conditions in Theorem 2.1 can be rephrased as the statement that joint ergodicity and w-joint ergodicity are equivalent. One can similarly show that uniform joint ergodicity and uniform w-joint ergodicity are equivalent, and can be characterized by a modified version of condition (3) in Theorem 2.1 (namely, with the averaging over $[0, N)$ in (3(b)) replaced by averaging over $[M, N)$ where $N - M \to \infty$). However, unlike the case of commuting transformations, joint ergodicity and uniform joint ergodicity are not equivalent (see Example 3.1).

REMARK 2.2. Let us explain why, in case $T_1, T_2, \ldots, T_s$ are invertible commuting transformations, condition (3(b)) in the theorem can be replaced by the requirement that each $T_iT_j^{-1}$, $1 \leqq i < j \leqq s$, be ergodic (compare with [2, Th. 3.1]). In fact, in this case (3(b)) is equivalent to $T_2T_1^{-1}, T_3T_1^{-1}, \ldots, T_sT_1^{-1}$ being (w-) jointly ergodic. Employing induction on the number s of transformations we observe that the latter condition is equivalent to the ergodicity of $(T_2T_1^{-1}) \times (T_3T_1^{-1}) \times \cdots \times (T_sT_1^{-1})$ and $T_iT_j^{-1}$, $2 \leqq i < j \leqq s$. From [2, Lemma 3.1] we now easily infer that (3(b)) can indeed be replaced with the condition in question. One might be tempted to ask now whether for non-commuting transformations (3(b)) can be replaced by the condition

$$\frac{1}{N}\sum_{n=0}^{N-1}\int_X T_i^n f \cdot T_j^n g\,d\mu \xrightarrow[N\to\infty]{} \int_X f\,d\mu \cdot \int_X g\,d\mu, \quad 1 \leqq i < j \leqq s, \quad f, g \in L^\infty(X, \mathscr{B}, \mu).$$

It follows, however, from [1, Ex. 4.1] that the resulting modified condition (3) is strictly weaker than joint ergodicity even for toral endomorphisms.

REMARK 2.3. For other applications in similar contexts of the van der Corput inequality, as well as of generaliztions thereof, see [3] and [4].

3. Endomorphisms of compact abelian groups; an example

Let G be a compact abelian group, $\mathscr{B}$ its Borel field, μ the Haar measure and Γ the dual group of G. An endomorphism of G is measure preserving iff it is onto. We shall now provide criteria for joint ergodicity and for uniform joint ergodicity of several epimorphisms of G, which are more convenient than those given in [1, Th. 4.1]. Recall that a set $S \subseteq \mathbf{N}$ is *of* 0 *density* if $\#(S \cap [1, N]) = o(N)$ and *of* 0 *Banach density* if $\#(S \cap [M, N)) = o(N - M)$ (where $\#(F)$ denotes the cardinality of a finite set F).

THEOREM 3.1. *Let $\sigma_1, \sigma_2, \ldots, \sigma_s$ be epimorphisms of G. $\sigma_1, \sigma_2, \ldots, \sigma_s$ are jointly ergodic (resp. uniformly jointly ergodic) iff each σ_i is ergodic and for every $\gamma_1, \gamma_2, \ldots, \gamma_s \in \Gamma$, not all of which are* 0, *the set $\{n : \Sigma_{i=1}^s \gamma_i \sigma_i^n = 0\}$ is of* 0 *density (resp. of* 0 *Banach density).*

The theorem follows easily from Theorem 2.1 and the fact that linear combinations of characters are dense in $C(G)$.

REMARK 3.1. It is easy to see that when applying the theorem it suffices to prove the 0 density (or 0 Banach density) of the sets $\{n : \Sigma_{i=1}^s \gamma_i \sigma_i^n = 0\}$ for s-tuples $(\gamma_1, \gamma_2, \ldots, \gamma_s)$ satisfying $\Sigma_{i=1}^s \gamma_i = 0$.

EXAMPLE 3.1. We shall construct a pair of jointly ergodic automorphisms of a compact abelian group which are not uniformly jointly ergodic. Let $G = C_2^{\mathbf{Z}}$, where $C_2 = \{0, 1\}$ is the cyclic group of order 2. The dual group Γ consists of all doubly-infinite sequences of 0's and 1's, having only finitely many non-zero components, the operation being addition modulo 2 componentwise. If $\gamma = (\ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots) \in \Gamma$ and $x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in G$, then $\gamma(x) = \Pi_{i=-\infty}^{\infty} (-1)^{\gamma_i x_i}$. Let σ be the left shift on G, that is

$$(\sigma x)_k = x_{k+1}, \qquad x \in G, \quad k \in \mathbf{Z}.$$

Let S be a subset of $\mathbf{Z}$, symmetric about 0, to be determined later. Define a permutation π on $\mathbf{Z}$ by

$$\pi(k) = \begin{cases} k, & k \in S, \\ -k, & k \notin S. \end{cases}$$

Let τ be the automorphism of G given by

$$(\tau x)_k = x_{\pi(\pi(k)+1)}.$$

We have

$$(\tau^n x)_k = x_{\pi(\pi(k)+n)}.$$

Obviously, σ and τ are ergodic. To ensure that σ, τ will be jointly ergodic we have to choose S so that for any $0 \neq \gamma \in \Gamma$ the set $\{n : \gamma\sigma^n = \gamma\tau^n\}$ will be of 0 density. It is easy to verify that the dual actions of σ and τ on Γ are given by

$$(\gamma\sigma)_k = \gamma_{k-1}, \qquad \gamma \in \Gamma, \quad k \in \mathbf{Z},$$

$$(\gamma\tau)_k = \gamma_{\pi(\pi(k)-1)}, \qquad \gamma \in \Gamma, \quad k \in \mathbf{Z}.$$

For any $\gamma \in \Gamma$ put

$$F_\gamma = \{k \in \mathbf{Z}\colon \gamma_k = 1\}, \qquad b_\gamma = \max\{|k|\colon k \in F_\gamma\}.$$

It is easy to check that

$$F_{\gamma\sigma^n} = F_\gamma + n, \qquad \gamma \in \Gamma, \quad n = 0, 1, 2, \dots$$

$$F_{\gamma\tau^n} = \pi(\pi(F_\gamma) + n), \qquad \gamma \in \Gamma, \quad n = 0, 1, 2, \dots$$

and so for $0 \neq \gamma \in \Gamma$ we have

$$F_{\gamma\sigma^n} \subseteq [-b_\gamma + n, b_\gamma + n], \qquad n = 0, 1, 2, \dots,$$

$$F_{\gamma\tau^n} \subseteq \pi([-b_\gamma + n, b_\gamma + n]), \quad n = 0, 1, 2, \dots .$$

If S is chosen as a set of 0 density, then for all n, with the exception of a set of 0 density, we have $F_{\gamma\tau^n} \subseteq [-b_\gamma - n, b_\gamma - n]$. It follows that $F_{\gamma\sigma^n} \cap F_{\gamma\tau^n} = \emptyset$ for all n outside a set of 0 density, and in particular the set $\{n\colon \gamma\sigma^n = \gamma\tau^n\}$ is of 0 density, so that σ, τ are jointly ergodic.

Now take

$$\gamma = (\dots, \overset{-2}{0}, \overset{-1}{0}, \overset{0}{1}, \overset{1}{0}, \overset{2}{0}, \dots).$$

We have $\gamma\sigma^n = \gamma\tau^n$ for every $n \in S$. Hence if S is not of 0 Banach density, then σ, τ are not uniformly jointly ergodic. Thus, taking S to be of 0 density but not of 0 Banach density (say, $S = \bigcup_{n=1}^{\infty} [n^3, n^3 + n]$) we obtain automorphisms σ and τ possessing the required properties.

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